RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 8 Solution

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1 Introduction

In the last two chapters, we would like to model asset dynamics and price options (or other derivatives). Recall that we are able to price a forward contract without assuming any model. This is not the case for option valuation. In this course, we will discuss two ways to model asset dynamics:

(1) **Binomial Lattice Model** (Discrete Time)

(2) **Brownian Motion** and **Stochastic Calculus** (Continuous Time)

2 Binomial Lattice Model

Setting:

- $S = S_0$ is known.
- After one period (Δt), S becomes uS (upward) or dS (downward)
- $u > 1$ and $0 < d < 1$.
- Probability of going up is p.
- Assume the following two quantities are known:
  (1) Expected annual rate of return: $\nu = \mathrm{E}[\log(S_{t+1}/S_t)]$.
  (2) Variance of the annual rate of return: $\sigma^2 = \mathrm{Var}[\log(S_{t+1}/S_t)]$.

  Using

$$\begin{aligned} v\Delta t &= \mathrm{E}(\log(S_{\Delta t})) \\ \sigma^2\Delta t &= \mathrm{Var}(\log(S_{\Delta t})) \\ u &= \frac{1}{d}. \end{aligned}$$

We find that

$$\begin{aligned} p &= \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{1+\frac{\sigma^2}{\nu^2\Delta t}}} \\ u &= \exp\{\sqrt{(v\Delta t)^2 + \sigma^2\Delta t}\} \\ d &= \exp\{-\sqrt{(v\Delta t)^2 + \sigma^2\Delta t}\}. \end{aligned}$$

For Δt small, we can use

$$
\begin{aligned}
p &= \frac{1}{2} + \frac{1}{2}\frac{\nu}{\sigma}\sqrt{\Delta t} \\
u &= e^{\sigma\sqrt{\Delta t}} \\
d &= e^{-\sigma\sqrt{\Delta t}}.
\end{aligned}
$$

Example 2.1. (2012-2013 Final Q4) Consider the price of a stock at time $t = 0$ given by $S_0 = 16.0$, with the mean and standard deviation of the annual rate of return given by $\nu = 10\%$ and $\sigma = 25\%$ respectively. Suppose that the risk-free rate is given by 5%.

(a) Based on $\Delta t = 1/12$ (i.e. one month), draw a 3−step binomial lattice for the next three months S_1, S_2 and S_3. What are the probabilities of obtaining each of the possible outcomes for S_3?.

(b) Estimate the price of a European Call option expiring in 2 months with strike $K = 16.0$ based on a binomial model.

Solution. (a) $u = 1.0748; d = 0.9304; p = 0.5722.$
$S_0 = 16; uS = 17.197; dS = 14.886; u^2S = 18.484; d^2S = 13.850; u^3S = 19.868; d^3S = 12.885.$
$P(S_3 = u^3S) = 0.1873; P(S_3 = uS) = 0.4202; P(S_3 = dS) = 0.3142; P(S_3 = d^3S) = 0.0783.$

(b) $p^* = 0.51095.$
$f = 0.6431.$

3 Lognormal Distribution

Definition 3.1. *If $X \sim N(\mu, \sigma^2)$, then the random variable $Y := e^X$ is **lognormally distributed**.*

It is clear that $Y \geq 0$.

Example 3.1. (2012-2013 Final Q3 Modified) Consider a random variable Y which is log-normally distributed, with $\log Y = X \sim N(\mu, \sigma^2)$.

(a) Show that the **moment generating function** $M_X(t)$ of X is $\exp(\mu t + \sigma^2 t^2/2)$.

(b) Show that the **variance** of Y is $\mathrm{Var}(Y) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$.

(c) Compute the **skewness** of Y, $\kappa_3 = \mathrm{E}([Y - \mu_Y]^3)/\sigma_Y^3$. Does it suggest that the distribution is right-skewed?

Remark 3.1. A distribution is right-skewed if the right tail is longer or fatter than the left side.

Solution. (a) To find the moment generating function or deal with normal density, we often use the completing square trick:

$$
\begin{aligned}
\mathrm{E}(e^{Xt}) &= \int_{-\infty}^{\infty} e^{xt}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2 - 2x\mu + \mu^2 - 2x\sigma^2 t}{2\sigma^2}\right\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[x-(\mu+\sigma^2 t)]^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right\}dx \\
&= \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right\}dx \\
&= \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}.
\end{aligned}
$$

(b)

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(e^X) \\
&= \mathrm{E}(e^X)^2 - [\mathrm{E}(e^X)]^2 \\
&= E(e^{2X}) - [\mathrm{E}(e^X)]^2 \\
&= e^{2\mu+2\sigma^2} - [e^{\mu+\sigma^2/2}]^2 \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\mathrm{E}(Y-\mu_Y)^3}{\sigma_Y^3} &= \frac{\mathrm{E}\,Y^3 - 3\mu_Y\,\mathrm{E}\,Y^2 + 3\mu_Y^2\,\mathrm{E}\,Y - \mu_Y^3}{(\exp(2\mu+\sigma^2)[\exp(\sigma^2)-1])^{3/2}} \\
&= \frac{\exp(3\mu+\frac{9}{2}\sigma^2) - 3\exp(3\mu+\frac{5}{2}\sigma^2) + 3\exp(3\mu+\frac{3}{2}\sigma^2) - \exp(3\mu+\frac{3}{2}\sigma^2)}{\exp(3\mu+\frac{3\sigma^2}{2})(\exp(\sigma^2)-1)^{3/2}} \\
&= \frac{\exp(3\sigma^2) - 3\exp(\sigma^2) + 2}{(\exp(\sigma^2)-1)^{3/2}} \\
&= \frac{(\exp(\sigma^2))^3 - 3\exp(\sigma^2) + 2}{(\exp(\sigma^2))^2 - 2\exp(\sigma^2) + 1}\sqrt{\exp(\sigma^2)-1} \\
&= (\exp(\sigma^2)+2)\sqrt{\exp(\sigma^2)-1} \\
&> 0.
\end{aligned}$$

Therefore, log-normal distribution is right-skewed.

Example 3.2 (09-10 Final Q3)**.** A simple gross return, $(1+R)$ is lognormal $(0,(0.1)^2)$, which means that $\log(1+R)$ is $N(0,(0.1)^2)$. Evaluate $P(1+R<0.9)$. You may leave your answer in terms of the CDF of a standard normal random variable.

Solution.

$$\begin{aligned}
P(1+R<0.9) &= P(\log(1+R) < \log(0.9)) \\
&= P(\frac{\log(1+R)}{0.1} < \frac{\log(0.9)}{0.1}) \\
&= \Phi(\frac{\log(0.9)}{0.1}) = \Phi(-1.0536),
\end{aligned}$$

where Φ is the CDF of a standard normal random variable.

4 Stochastic Processes and Martingales

A **stochastic process** X_t (or $X(\omega,t)$) is a collection of random variables indexed by an index set T. It is a function from $\Omega \times T \to \mathbb{R}$, where Ω is the sample space. Usually, $T = \{0,1,2,\ldots\}$ (which corresponds to a **discrete stochastic process**) or $T = [0,\infty)$ (which corresponds to a **continuous stochastic process**).

We can think of t as time and each ω is a particle. With this picture in mind, $X_t(\omega)$ would represent the position at time t of the particle (experiment) ω.

Definition 4.1. *A stochastic process $\{X_n; n \in \mathbb{N}\}$ is a* ***martingale*** *if for each n,*

(a) $\mathrm{E}(|X_n|) < \infty$

(b) $\mathrm{E}(X_n \mid X_1, \ldots, X_{n-1}) = X_{n-1}$

Remark 4.1. If we think of $X_n - X_{n-1}$ as the net winnings per unit stake in game n in a series of game, played at time $n = 1, 2, \ldots$. Then, we have $\mathrm{E}(X_n - X_{n-1} \mid X_1, \ldots, X_{n-1}) = 0$. So that the game series is fair.

Example 4.1. Let X_n be independent random variables with $\mathrm{E}(X_n) = 0$ for all $n \geq 1$, then $S_n = X_1 + \cdots + X_n$ is a martingale.

Solution.

$$\begin{aligned}
\mathrm{E}(S_n \mid S_1, \ldots, S_{n-1}) &= \mathrm{E}(S_{n-1} + X_n \mid S_1, \ldots, S_{n-1}) \\
&= S_{n-1} + \mathrm{E}(X_n) \\
&= S_{n-1}.
\end{aligned}$$

Example 4.2. Let X_n be a symmetric random walk, that is $X_n = Z_1 + \cdots + Z_n$ where Z_i are iid random variable with $P(Z_n = 1) = P(Z_n = -1) = 1/2$. Show that $X_n^2 - n$ is a martingale.
(Hint: Use the properties of conditional expectation.)

Solution.

$$\begin{aligned}
\mathrm{E}(X_{n+1}^2 - (n+1) \mid X_1, \ldots, X_n) &= \mathrm{E}((X_n + Z_{n+1})^2 - (n+1) \mid X_1, \ldots, X_n) \\
&= \mathrm{E}(X_n^2 + 2X_n Z_{n+1} + Z_{n+1}^2 - (n+1) \mid X_1, \ldots, X_n) \\
&= X_n^2 + 2X_n \,\mathrm{E}(Z_{n+1}) + \mathrm{E}(Z_{n+1}^2) - (n+1) \\
&= X_n^2 + 0 + 1 - (n+1) \\
&= X_n^2 - n.
\end{aligned}$$

5 Wiener Process

Definition 5.1. *A stochastic process $W(t)$ is called a standard Brownian motion or Wiener process if*

(1) With probability one, $W(0) = 0$.

(2) ($W(t)$ has independent increments) For any $t_1 < t_2 \leq t_3 < t_4$, $W(t_4) - W(t_3)$ is independent with $W(t_2) - W(t_1)$.

(3) For $s < t$, $W(t) - W(s) \sim N(0, t - s)$.

(4) Almost all sample paths of W_t are continuous functions.

Properties:

- $W(t)$ is normally distributed for any t.
- For $s < t$,
$$\mathrm{E}(W_t \mid W_s) = W_s.$$
- $W(t)$ is nowhere differentiable (with probability one).

Remark 5.1. A formal introduction of Brownian motion, conditional expectation, martingale, Ito integral and stochastic differential equation will be given in RMSC4005 Stochastic Calculus.

Example 5.1. Let W_t be a Brownian motion. Show that for $s < t$, $W_t^2 - t$ satisfies

$$\mathrm{E}(W_t^2 - t \mid W_s) = W_s^2 - s.$$

What can you conclude for $W_t^2 - t$?

Solution.

$$\begin{aligned}
\mathrm{E}(W_t^2 - t \mid W_s) &= \mathrm{E}[(W_t - W_s + W_s)^2 | W_s] - t \\
&= \mathrm{E}[(W_t - W_s)^2 + 2W_s(W_t - W_s) + W_s^2 | W_s] - t \\
&= \mathrm{E}[(W_t - W_s)^2] + 2W_s \,\mathrm{E}(W_t - W_s) + W_s^2 - t \\
&= (t - s) + W_s^2 - t \\
&= W_s^2 - s.
\end{aligned}$$

Therefore, $W_t^2 - t$ is a martingale.

Example 5.2. Prove that $W_t^3 - 3tW_t$ is a martingale.

Solution.

$$\begin{aligned}
& \mathrm{E}(W_t^3 - 3tW_t \mid W_s) \\
= {} & \mathrm{E}[(W_t - W_s + W_s)^3 - 3t(W_t - W_s + W_s) | W_s] \\
= {} & \mathrm{E}[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3 - 3t(W_t - W_s) | W_s] - 3tW_s \\
= {} & \mathrm{E}(W_t - W_s)^3 + 3W_s \,\mathrm{E}(W_t - W_s)^2 + 3W_s^2 \,\mathrm{E}(W_t - W_s) + W_s^3 - 3t \,\mathrm{E}(W_t - W_s) - 3tW_s \\
= {} & 3W_s(t - s) + W_s^3 - 3tW_s \\
= {} & W_s^3 - 3sW_s.
\end{aligned}$$

Remark 5.2. Some properties of conditional expectations:

(1) (Independence) $\mathrm{E}(g(X)|Y) = \mathrm{E}(g(X))$ if X and Y are independent.

(2) (Taking out what is known) $\mathrm{E}(g(X)h(Y)|Y) = h(Y)\,\mathrm{E}(g(X)|Y)$.

6 Appendix

This appendix summarizes some of the facts about option you learned in RMSC2001.

6.1 Options

Definition 6.1. *An* ***option*** *is a contract which gives the buyer the* ***right*** *(but not the obligation) to buy or sell an underlying asset at a specified strike price on or before a specified date.*

Two basic types:

(1) A **call option** gives the holder of the option the right to buy an asset by a certain date for a certain price.

(2) A **put option** gives the holder the right to sell an asset by a certain date for a certain price.

Terms:

(1) **Maturity date**: the date specified in the contract

(2) **Exercise price/ Strike price**: the price specified in the contract

Payoff of the options: (Note that payoff $\neq$ profit. Profit is payoff minus cost)

(1) Call option: $\max(S_T - K, 0)$.

(2) Put option: $\max(K - S_T, 0)$.

where T is the maturity date and K is the strike price. According to when you can exercise:

(1) **European option**: an option that may only be exercised on expiration

(2) **American option**: an option that may be exercised on any trading day on or before expiry.

(3) **Bermudan option**: an option that may be exercised only on specified dates on or before expiration. (Bermuda is between Europe and America!)

If the structure of the option is different from standard calls and puts (**plain vanilla products**), then it is referred to as an **exotic option**. You may learn about how to use Monte Carlo simulation to price exotic options in RMSC4001. Examples of exotic options may include asian option, binary option, barrier option, etc.

Example 6.1. Forward start options: options that will start at some time in the future. Sometimes employee stock options (about corporate finance) can be viewed as forward start options. It is common to specify that the strike price will be set in the future so that the option is initially at the money or a certain percentage in the money or out of the money.

Example 6.2. The payoffs from **lookback options** depend on the maximum or minimum asset price reached during the life of the option. Payoff:

(i) Floating lookback call: $\max\{0, S_T - \min_t S_t\}$.

(ii) Floating lookback put: $\max\{0, \max_t S_t - S_T\}$.

6.2 Risk-Neutral Valuation

(a) Let Π be the value of the portfolio consisting a long positing in Δ shares and a short position in one option (note that no specific option is assumed!). Therefore, at the end (time T) of the life of the option if there is an upward movement in the stock price, $\Pi_T = S_0 u\Delta - v_u$. Similarly, if there is a downward movement in the stock price, $\Pi_T = S_0 d\Delta - v_d$.

(b) To make the portfolio riskless, its value at maturity must be constant. That is,

$$S_0 u\Delta - v_u = S_0 d\Delta - v_d.$$

Rearranging the terms, we have

$$\Delta = \frac{v_u - v_d}{S_0 u - S_0 d}.$$

(c) Since the portfolio is riskless, it must earn the risk-free rate r. Therefore, $\Pi_0 = \Pi_T e^{-rT}$. That is,

$$S_0\Delta - v = (S_0 u\Delta - v_u)e^{-rT}.$$

Rearranging, we have

$$v = S_0\Delta - (S_0 u\Delta - v_u)e^{-rT}. \tag{1}$$

(d) By direct substitution of $\Delta = \dfrac{v_u - v_d}{S_0u - S_0d}$ into (1). We have,

$$\begin{aligned} v &= \frac{v_u - v_d}{u-d} - \left(\frac{u(v_u - v_d)}{u-d} - v_u\right)e^{-rT} \\ &= e^{-rT}\left[e^{rT}\frac{v_u - v_d}{u-d} - \frac{dv_u - uv_d}{u-d}\right] \\ &= e^{-rT}\left[\frac{e^{rT}-d}{u-d}v_u + \left(1 - \frac{e^{rT}-d}{u-d}\right)v_d\right]. \end{aligned}$$

If we denote $p^* := \dfrac{e^{rT}-d}{u-d}$, then we have

$$v = e^{-rT}[p^* v_u + (1-p^*)v_d].$$

If there is no arbitrage, then $p^* \in (0,1)$ (otherwise, buying a risk-free asset can always dominate the payoff of a stock or buying the stock can always dominate the payoff of a risk-free asset). Therefore, it is natural to interpret p^* as the probability of the up movement. Under this probability,

$$\mathrm{E}^*(S_T) = p^* S_0 u + (1-p^*)S_0 d = p^* S_0(u-d) + S_0 d = S_0 e^{rT}.$$

Therefore, under this probability, the expected return on the stock equals the risk-free rate. This is an example of risk-neutral valuation and you will learn more in other RMSC courses.

Remark 6.1. Note that no assumption on the payoff of the option is needed. That means, you can price any option in this way.

6.3 Multi-period Binomial Trees

With two periods, we have

$$\begin{aligned} v_u &= e^{-r\Delta t}[p^* v_{uu} + (1-p^*)v_{ud}] \\ v_d &= e^{-r\Delta t}[p^* v_{du} + (1-p^*)v_{dd}] \\ v &= e^{-r\Delta t}[p^* v_u + (1-p^*)v_d] \\ p^* &= \frac{e^{r\Delta t}-d}{u-d}. \end{aligned}$$

After substitution, we have

$$v = e^{-2r\Delta t}[(p^*)^2 v_{uu} + p^*(1-p^*)v_{ud} + p^*(1-p^*)v_{du} + (1-p^*)^2 v_{dd}].$$

Here p^* is the risk-neutral probability. v_{uu} is the payoff when u^2S occurs. Other notations are defined similarly. For simplicity, I assumed u and d are the same at each node of the tree and so that p^* is the same at each node.

6.4 Examples

Example 6.3 (Two-step Binomial Tree on Exotic Options)**.** Consider a two-step binomial tree for the stock price $S(t)$ at $t = 0, 1, 2$ year. The stock price starts at $S(0) = 100$ and moves either up or down by 20% per year. Assume that the (continuously compounded) risk-free rate is 3% per annum.

(a) Estimate the price of a European put option with strike $K = 110$ and expiry $= 2$ years.

(b) Estimate the price of a European option with expiry at 2-year and the payoff function given by:

$$\text{Payoff} = \max\{S(1) - 110, 0\} + \max\{S(2) - 110, 0\}$$

Solution. (a) First, calculate p^*,

$$p^* = \frac{e^{0.03} - 0.8}{1.2 - 0.8} = 0.57614.$$

The payoffs of the put option when stock price is S_{uu}, S_{ud} and S_{dd} are $0, 14$ and 46 respectively.

$$v = [0(p^*)^2 + 14(2)(p^*)(1 - p^*) + 46(1 - p^*)^2]e^{-0.03 \cdot 2} = 14.2226.$$

(b) Note that $v_{uu} = 44, v_{ud} = 10, v_{du} = 0$ and $v_{dd} = 0$.

$$v = [44(p^*)^2 + 10(p^*)(1 - p^*) + 0(p^*)(1 - p^*) + 0(1 - p^*)^2]e^{-0.03 \cdot 2} = 16.0545.$$